# Chapter 3

X - discrete RV:
pmf - $P(X = x) = f_X(X)$
$E[X] = \sum_{x \in S} xP(X = x)$
$E[X^n]$ =nth moment
$Var[X] = E[(X - E[X])^2]$

X - continuous RV:
pdf - $f_X(x) = P(x \leq X \leq x + dx)$
$E[X] = \int_{-\infty}^{\infty} x f_X(x)dx$
$E[X^n] = \int -\infty^{\infty} x^n f_X(x)dx$

Relationships:
$P(A|B) = \frac{P(A \cap B)}{P(B)}$
Bayes: $P(A) = P(A|B)P(B) + P(A|B^C)P(B^C)$
discrete - $P_{X|Y}(x, y) = P(X = x|Y = y) = \frac{P(X=x,Y=y)}{P(Y=y)}$
continuous - $F_{X|Y}(x, y) = P(X \leq x|Y = y) = \sum_{a \leq X} P_{X|Y}(x|y)$
$E[X|Y = y] = \sum_x xP(X = x|Y = y) = \sum_x xP_{X|Y}(x|y)$
expectation gives RV - $E[X|Y] = g(Y)$

X $\perp$ Y:
$P_{X|Y}(x|y) = P_X(x)$
$F_{X|Y}(x|y) = F_X(x)$
$E[X|Y = y] = E[X] \quad \forall y$

X,Y continuous, $f_X(x), f_Y(y)$, joint $f_{XY}(x, y)$:
$f_{X|Y}(X|Y) = \frac{f_{XY}(x,y)}{f_Y(Y)}$
$E[X|Y = y] = \int x f_{X|Y}(x, y)dx$

Law of Total Expectation:
discrete - $E[X] = \sum_y E[X|Y = y]P(Y = y)$
continuous - $E[X] = \int E[X|Y = y]f_Y(y)dy$
general - $P(A) = \sum_y P(A|Y = y)$
law - $E[E[X|Y]] = E[X]$

# Chapter 4

## Discrete Time Markov Chains

Markov Property:
$P(X_{n+1} = j|X_n = i_n, \ldots, X_0 = i_0) = P(X_{n+1} = j|X_n = i_n)$
Markov Property V2:
$P(X_{n+m} = j|X_n = i, \ldots, X_0 = i_0) = P(X_{n+m} = j|X_n = i)$
Time Homogeneous:
$P(X_{n+m} = j|X_m = i) = P(X_{n+m+k} = j|X_{m+k} = i)$
Transition Probabilities:
$P_{ij} = P(X_{n+1} = j|X_n = i) = P(X_1 = j|X_0 = i_n)$
k-step - $P^k_{ij} = P(X_{n+k} = j|X_n = i) = P(X_k = j|X_0 = i_n)$

Chapman-Kolmogorov:
$P(n + m) = P(n)P(m)$
$P(n) = P(1)^n$
$\therefore P_{ij} = P(X_{n+1} = j|X_0 = i) = \sum_s P^n_{sj} P_{is} =$
$= (i, j)^{th}\ entry\ of\ P(1)P(n)$
* the transition prob matrix is stochastic - $\sum_{j \in S} P_{ij} = 1 \quad \forall i \in S$

Classification of States:
j is *accessible* from i if $P^n_{ij} = (P(X_n = i|X_0 = j) > 0$, for $n > 0$
2 states *communicate* ($i \leftrightarrow j$)if they can access each other
If 2 states communicate they are in the same *class*.
Any 2 classes are identical or disjoint.
A MC is *irreducible* if it has only one class.

Recurrence/Transience:
Define $N(i)$ to be the number of times we visit state i
Define $f_i$ to be the probability of entering i, given we start in i
State i is: recurrent: if $f_i = 1$ ($N(i) = \infty$)
transient: if $f_i < 1$
Recurrence and transience are *class properties*
At least one state *must* be recurrent.
A finite state-space irreducible MC is recurrent.
recurrent:
$P(N(i) \geq k|X_0 = i) = (f_i)^k = 1 \iff \lim_{k \to \infty} P(N(i) \geq k|X_0 = 1)$
$\Rightarrow P(N(i) = \infty|X_0 = i) = 1$
$\iff \sum_{n=1}^{\infty} P^n_{ii} = \infty$
transient:
$\lim_{k \to \infty} P(N(i) \geq k|X_0 = i) = 0 \Rightarrow P(N(i) = \infty|X_0 = i) = 0$
$\iff \sum_{n=1}^{\infty} P^n_{ii} < \infty$

A state is *positive recurrent* if for
$\mathcal{C} = min\{n \geq 1|X_n = j\}$, $E[\mathcal{C}_j|X_0 = j] < \infty$
A state is *null recurrent* if $E[\mathcal{C}_j|X_0 = j] = \infty$
For finite state-space MC, recurrent $\Rightarrow$ positive recurrent

Period:
*Period* of a state $d = gcd(n|P^n_{ii} > 0)$
$P^n_{ii} > 0 \iff d|n$
$P^n_{ii} = 0 \iff d \nmid n$
A state is *aperiodic* if its period is 1.
Period is a class property
A MC is *ergodic* if it is positive recurrent and aperiodic.

Stationary Distbn / Generating Fns:
$\{X_n\}_{n \geq 0}$ - positive-recurrent MC, with stationary distbn $\{\pi_{ij}\}_{j \in S}$
$T_j = inf\{n \geq 1|X_n = j\}$ $\qquad m_{ij} = E[T_j|X_0 = j] \Rightarrow m_{jj}\pi_j = 1$
Recurrent MC with stationary distbn is actually *positive-recurrent*
$\pi_j m_{jj} = 1 \Rightarrow \pi_j \neq 0 \Rightarrow m_{jj} = \frac{1}{\pi_j} < \infty$
A MC is *reversible* wrt a distribution $\{\pi_i\}_{i \in S} \iff \pi_i P_{ij} = \pi_j P_{ji}$
If a MC is irreducible, aperiodic, and has a stationary distbn $\{\pi_i\}_{i \in S}$
then: $lim_{n \to \infty} P^n_{ij} = \pi_j \forall j \in S$
Note: $\sum_{j=0}^{\infty} \pi_j = 1$

Reversibility: Take $\{X_n\}_{n \geq 0}$ a MC w. stationary distbn
$\{\pi_i\}_{i \in S}$ - long-run proportion of transitions *out* of state i
$P_{ij}$ - probability of going from i to j
$\pi_i P_{ij}$ - long-run proportion of transitions from i to j
$\pi_j = \sum_{i \in S} \pi_i P_{ij}$ - *Global Balance Equation*
$\pi_j P_{ji} = \pi_i P_{ij}$ - *Local Balance Equation*

Reversible MC:
A MC is *reversible* wrt a stationary distbn if the Local Balance Eqn holds.
If a MC is reversible wrt $\{\pi_i\}_{i \in S}$ then $\{\pi_i\}_{i \in S}$ is a stationary distbn.
If $\{X_n\}_{n \in \mathbb{Z}} = \{\ldots, X_{-n}, X_{-n+1}, \ldots, X_0, \ldots, X_n, \ldots\}$ is a MC then $\{Y_n\}\ with\ Y_n = X_{-n}$ is a MC, and is called the *reversed process*

# Chapter 5

## Exponential Distribution

$f(x) = \lambda e^{-\lambda x}$
$F(x) = \int_{-\infty}^{x} f(x)dx = 1 - e^{-\lambda x}$

## Memoryless RV

$P\{X > s + t|X > t\} = P\{X > s\}$

## Gamma Distribution

$X_i$ drawn idd $\sim exp(\lambda) \Rightarrow X_1 + \ldots + X_n$is
gamma$(n, \lambda) f(t) = \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}$

## Comparing Exponentials

$P\{X_1 < X_2\} = \int_0^{\infty} P\{X_1 < X_2|X_1 = x\}\lambda_1 e^{-\lambda_1 x}dx =$
$\int_0^{\infty} P\{x < X_2\}\lambda_1 e^{-\lambda_1 x}dx = \frac{\lambda_1}{\lambda_1 + \lambda_2}$

## Counting Process

$\{N(t), t \geq 0\}$ is a counting process if:

- $N(t) > 0$
- $N(t)$ is integer valued
- If $s < t$ then $N(s) \leq N(t)$
- For $s < t$, $N(t) - N(s)$ equals the number of events that occur in the interval $(s, t]$

## Independent Increments

If the numbers of events that occur in the disjoint time intervals are independent

## Poisson Process

A counting process $\{N(t), t \geq 0\}$ is a Poisson process with rate $\lambda$ if:

- $N(0) = 0$
- $N(t)$ has independent increments
- Number of evens in a time interval of length t is Poisson with $\lambda$:
  $P\{N(t + s) - N(s) = n\} = e^{-\lambda t} \frac{(\lambda t)^n}{n!}, n = 0, 1, \ldots$

## $o(h)$ Functions

A function $f()$ is $o(h)$ if
$\lim_{h \to \infty} \frac{f(h)}{h} = 0$

## Poisson Process 2

- $N(0) = 0$
- The process has stationary and independent increments
- $P\{N(h) = 1\} = \lambda h + o(h)$
- $P\{N(h) \geq 2\} = o(h)$

## Interarrival Time

$T_n$is the time between the $(n - 1)$st and $n$th events. $\{T_n\}$ is the **sequence of internarrival times** with $T_n \sim \exp(\lambda)$

## Waiting Time

$S_n = \sum_{i=1}^{n} T_i \sim$gamma(n,$\lambda$)

# Chapter 6

## Continuous Time Markkov Chain

$\{X(t), t \geq 0\} \forall s, t \geq 0$, non-neg ints $i, j, x(u), 0 \leq u < s$ has
$P\{X(t + s) = j|X(s) = i, X(u) = x(u), 0 \leq u \leq s\}$
$= P\{X(t + s) = j|X(s) = i\}$

## Stationary Transition Probabilities

A CTMC has these if $P\{X(t + s) = j|X(s) = i\}$

## CTMC Alternate Definition

A stochastic process having these properties each time it enters state i:

- The amount of time it spends in that state before make a transition into a different state $\sim exp(\lambda)$ with mean $1/v_i$
- when the process leaves state i, it enters state j with some probability $P_{ij}$ and
  $P_{ii} = 0$ all i
  $\sum_j P_{ij} = 1$, all i

## Birth and Death Processes

A system with n people with

- $\{\lambda_n\}_{n=0}^{\infty}$ the arrival/birth rate
- $\{\mu_n\}_{n=0}^{\infty}$ the departure/death rate
- $v_0 = \lambda_0$
- $v_i = \lambda_i + \mu_i$
- $P_{01} = 1$
- $P_{i,i+1} = \frac{\lambda_i}{\lambda_i + \mu_i}$
- $P_{i,i-1} = \frac{\mu_i}{\lambda_i + \mu_i}$

## Transition Probability Function

Move from state i to state j in a time t later

## Instantaneous Transition Rates

$P_{ij}(t) = P\{X(t + s) = j|X(s) = i\}$
$q_{ij} = v_i P_{ij}$ is the rate, when in state i, at which the process makes a transition into state j.
$v_i = \sum_j v_i P_{ij} = \sum_j q_{ij}$
$P_{ij} = \frac{q_{ij}}{v_i} = \frac{q_{ij}}{\sum_j q_{ij}}$

## Chapman-Kolmogorov Equations

$P_{ij}(t + s) = \sum_{k=0}^{\infty} P_{ik}(t)P_{kj}(s), \forall s, t \geq 0$

## Kolmogorov's Backward Equations

$P'_{ij}(t) = \sum_{k \neq i} q_{ik}P_{kj}(t) - v_i P_{ij}(t)$

## Kolmogorov's Forward Equations

$P'_{ij}(t) = \sum_{k \neq j} q_{kj}P_{ik}(t) - v_j P_{ij}(t)$

## Limiting Probabilities

$P_j \equiv \lim_{t\to\infty} P_{ij}(t)$ The limiting probs will exist if

- all states communicate
- the chain is positive recurrent

Use that to get:
$v_j P_i = \sum_{k\neq j} q_{kj} P_k$, all states j
(leaving = entering)
$\sum_j P_j = 1$

## Embeded Chains

A ergodic CTMC has an embeded discrete-time ergodic MC with $P_{ij}$ and limiting probabilities $\pi_i$.

$P_i = \frac{\pi_i/v_i}{\sum_j \pi_j/v_j}$
$\pi_i = \sum_j \pi_j P_{ji}$

## Time Reversability

For a long running MC, the amount of time the process spends in state $i$ is also exponentially distributed with rate $v_i$. We have the discrete time reversed chain:
$Q_{ij} = \frac{\pi_j P_{ji}}{\pi_i}$ and then
$\pi_i P_{ij} = \pi_j P_{ji} \forall i,j$
$P_i q_{ij} = P_j q_{ji} \forall i,j$

## Computing the Transition Probabilities

need this?

# Chapter 8

## Definitions

| | |
|---|---|
| $L$ | average # of customers in system |
| $L_Q$ | average # of customers waiting in queue |
| $W$ | average amount of time a customer spends in system |
| $W_Q$ | average amount of time a customer spends in queue |
| $E[S]$ | average amount of time customer spends in service |
| $N(t)$ | number of customer arrivals by time t |
| $P_n$ | number of customers in system at time t<br>$= \lim_{t\to\infty} P\{X(t) = n\}$<br>• aka limiting/longrun/steady state probability that n customers are in the system<br>• also long run proportion of time that the system contains n customers |
| $\lambda_a$ | average arrival rate of customers<br>$= \lim_{t\to\infty} \frac{N(t)}{t}$ |
| $a_n$ | proportion of custs that find n in the system when arriving |
| $d_n$ | proportion of custs that find n in the system when leaving |

## Littile's Formula

$L = \lambda_a W$ $L_Q = \lambda_a W_Q$

## Poisson Model

Poisson arrivals see time averages. ie $P_n = a_n$.

## M/M/1

Customers arrive according to a Poisson process with rate $\lambda$. The time between successive arrivals are independent rv with mean $1/\lambda$. If server free, cust goes in, else into queue. Service times are $\sim\exp(\mu)$.
Balance Equations:
$\lambda P_0 = \mu P_1$
$(\lambda + \mu)P_n = \lambda P_{n-1} + \mu P_{n+1}$
$P_1 = \frac{\lambda}{\mu} P_0$
$P_{n+1} = \frac{\lambda}{\mu} P_n + (P_n - \frac{\lambda}{\mu} P_{n-1}) = \left(\frac{\lambda}{\mu}\right)^{n+1} P_0$
$1 = \sum_{n=0}^{\infty} P_n = \sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n P_0 = \frac{P_o}{1-\lambda/\mu}$ or
$\Rightarrow P_0 = 1 - \frac{\lambda}{\mu}, P_n = \left(\frac{\lambda}{\mu}\right)^n (1 - \frac{\lambda}{\mu})$
$L = \sum_{n=0}^{\infty} n P_n = \frac{\lambda}{\mu-\lambda}$
$W = \frac{L}{\lambda} = \frac{1}{\mu-\lambda}$
$W_Q = W - E[S] = W - 1/\mu = \frac{\lambda}{\mu(\mu-\lambda)}$
$L_Q = \lambda W_Q = \frac{\lambda^2}{\mu(\mu-\lambda)}$

## M/M/1 - Finite Capacity

Now we have the limitation that $n \leq N$.
Balance Equations:
$\lambda P_0 = \mu P_1$
$(\lambda + \mu)P_n = \lambda P_{n-1} + \mu P_{n+1}, 1 \leq n \leq N-1$
$\mu P_n = \lambda P_{n-1}$ , for state N
$P_1 = \frac{\lambda}{\mu} P_0$
$P_{n+1} = \frac{\lambda}{\mu} P_n + (P_n - \frac{\lambda}{\mu} P_{n-1}), 1 \leq n \leq N-1$
$P_N = \frac{\lambda}{\mu} P_{N-1} \cdots = \left(\frac{\lambda}{\mu}\right)^N P_0$
$1 = \sum_{n=0}^{\infty} P_n = P_0 \sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n = P_0 \frac{1-(\lambda/\mu)^{N+1}}{1-\lambda/\mu}$ or
$\Rightarrow P_0 = \frac{1-\lambda/\mu}{1-(\lambda/\mu)^{N+1}}, P_n = \frac{(\lambda/\mu)^n(1-\lambda/\mu)}{1-(\lambda/\mu)^{N+1}}, n = 0,\ldots,N$
$L = \sum_{n=0}^{N} n P_n = \frac{(1-\lambda/\mu)}{1-(\lambda/\mu)^{N+1}} \sum_{n=0}^{N} n \left(\frac{\lambda}{\mu}\right)^n =$
$= \frac{\lambda[1+N(\lambda/\mu)^{N+1} - (N+1)(\lambda/\mu)^N]}{(\mu-\lambda)(1-(\lambda/\mu)^{N+1})}$
To find W we consider 2 cases.
$\lambda_a = \lambda$ if "customers in system" includes those who never get in
$\lambda_a = \lambda(1 - P_N)$ if it does not. Either way we get:
$W = \frac{L}{\lambda_a}$

## PASTA

Poisson Arrivals See Time Averages
Let $\{X_i\}_{t\geq 0}$ a continuous-time Markov chain with stationary distribution $\pi$. Let $T_i$ be the arrival time of the $i^{th}$ element. These elements arrive according to a Poisson process.
Then $\{X(T_n)\}_{t\geq 0}$ has $\pi$ as a stationary distribution.
$a_j = \pi_j$

## Other useful stuff

$\sum_{n=0}^{\infty} n x^n = \frac{x}{(1-x)^2}$