Chapter 3

X - discrete RV:
pmf - $P(X=x) = f_X(X)$
$E[X] = \sum_{x \in S} xP(X=x)$
$E[X^n]$ =nth moment
$Var[X] = E[(X - E[X])^2]$

X - continuous RV:
pdf - $f_X(x) = P(x \leq X \leq x + dx)$
$E[X] = \int_{-\infty}^{\infty} xf_X(x)dx$
$E[X^n] = \int -\infty^{\infty} x^n f_X(x)dx$

Relationships:
$P(A|B) = \frac{P(A \cap B)}{P(B)}$
Bayes: $P(A) = P(A|B)P(B) + P(A|B^C)P(B^C)$
discrete - $P_{X|Y}(x,y) = P(X=x|Y=y) = \frac{P(X=x, Y=y)}{P(Y=y)}$
continuous - $F_{X|Y}(x,y) = P(X \leq x|Y=y) = \sum_{a \leq X} P_{X|Y}(x|y)$
$E[X|Y=y] = \sum_x xP(X=x|Y=y) = \sum_x xP_{X|Y}(x|y)$
expectation gives RV - $E[X|Y] = g(Y)$

X $\perp$ Y:
$P_{X|Y}(x|y) = P_X(x)$
$F_{X|Y}(x|y) = F_X(x)$
$E[X|Y=y] = E[X] \quad \forall y$

X,Y continuous, $f_X(x), f_Y(y)$, joint $f_{XY}(x,y)$:
$f_{X|Y}(X|Y) = \frac{f_{XY}(x,y)}{f_Y(Y)}$
$E[X|Y=y] = \int xf_{X|Y}(x,y)dx$

Law of Total Expectation:
discrete - $E[X] = \sum_y E[X|Y=y]P(Y=y)$
continuous - $E[X] = \int E[X|Y=y]f_Y(y)dy$
general - $P(A) = \sum_y P(A|Y=y)$
law - $E[E[X|Y]] = E[X]$

Chapter 4

Discrete Time Markov Chains

Markov Property:
$P(X_{n+1} = j|X_n = i_n, \ldots, X_0 = i_0) = P(X_{n+1} = j|X_n = i_n)$
Markov Property V2:
$P(X_{n+m} = j|X_n = i, \ldots, X_0 = i_0) = P(X_{n+m} = j|X_n = i)$
Time Homogeneous:
$P(X_{n+m} = j|X_m = i) = P(X_{n+m+k} = j|X_{m+k} = i)$
Transition Probabilities:
$P_{ij} = P(X_{n+1} = j|X_n = i) = P(X_1 = j|X_0 = i_n)$
k-step - $P_{ij}^k = P(X_{n+k} = j|X_n = i) = P(X_k = j|X_0 = i_n)$

Chapman-Kolmogorov:
$P(n+m) = P(n)P(m)$
$P(n) = P(1)^n$
$\therefore P_{ij} = P(X_{n+1} = j|X_0 = i) = \sum_s P_{sj}^n P_{is} =$
$= (i,j)^{th}\ entry\ of\ P(1)P(n)$
* the transition prob matrix is stochastic - $\sum_{j \in S} P_{ij} = 1 \quad \forall i \in S$

Classification of States:
j is *accessible* from i if $P_{ij}^n = (P(X_n = i|X_0 = j) > 0$, for $n > 0$
2 states *communicate* $(i \leftrightarrow j)$if they can access each other
If 2 states communicate they are in the same *class*.
Any 2 classes are identical or disjoint.
A MC is *irreducible* if it has only one class.

Recurrence/Transience:
Define $N(i)$ to be the number of times we visit state i
Define f_i to be the probability of entering i, given we start in i
State i is: recurrent: if $f_i = 1$ $(N(i) = \infty)$
transient: if $f_i < 1$
Recurrence and transience are *class properties*
At least one state *must* be recurrent.
A finite state-space irreducible MC is recurrent.
recurrent:
$P(N(i) \geq k|X_0 = i) = (f_i)^k = 1 \iff \lim_{k \to \infty} P(N(i) \geq k|X_0 = 1)$
$\Rightarrow P(N(i) = \infty|X_0 = i) = 1$
$\iff \sum_{n=1}^{\infty} P_{ii}^n = \infty$
transient:
$\lim_{k \to \infty} P(N(i) \geq k|X_0 = i) = 0 \Rightarrow P(N(i) = \infty|X_0 = i) = 0$
$\iff \sum_{n=1}^{\infty} P_{ii}^n < \infty$

A state is *positive recurrent* if for
$\mathcal{C} = min\{n \geq 1|X_n = j\}, E[\mathcal{C}_j|X_0 = j] < \infty$
A state is *null recurrent* if $E[\mathcal{C}_j|X_0 = j] = \infty$
For finite state-space MC, recurrent $\Rightarrow$ positive recurrent

Period:
Period of a state $d = gcd(n|P_{ii}^n > 0)$
$P_{ii}^n > 0 \iff d|n$
$P_{ii}^n = 0 \iff d \nmid n$
A state is *aperiodic* if its period is 1.
Period is a class property
A MC is *ergodic* if it is positive recurrent and aperiodic.

Stationary Distbn / Generating Fns:
$\{X_n\}_{n \geq 0}$ - positive-recurrent MC, with stationary distbn $\{\pi_{ij}\}_{j \in S}$
$T_j = inf\{n \geq 1|X_n = j\} \qquad m_{ij} = E[T_j|X_0 = j] \Rightarrow m_{jj}\pi_j = 1$
Recurrent MC with stationary distbn is actually *positive-recurrent*
$\pi_j m_{jj} = 1 \Rightarrow \pi_j \neq 0 \Rightarrow m_{jj} = \frac{1}{\pi_j} < \infty$
A MC is *reversible* wrt a distribution $\{\pi_i\}_{i \in S} \iff \pi_i P_{ij} = \pi_j P_{ji}$
If a MC is irreducible, aperiodic, and has a stationary distbn $\{\pi_i\}_{i \in S}$
then: $lim_{n \to \infty} P_{ij}^n = \pi_j \forall j \in S$
Note: $\sum_{j=0}^{\infty} \pi_j = 1$

Reversibility: Take $\{X_n\}_{n \geq 0}$ a MC w. stationary distbn
$\{\pi_i\}_{i \in S}$ - long-run proportion of transitions *out* of state i
P_{ij} - probability of going from i to j
$\pi_i P_{ij}$ - long-run proportion of transitions from i to j
$\pi_j = \sum_{i \in S} \pi_i P_{ij}$ - *Global Balance Equation*
$\pi_j P_{ji} = \pi_i P_{ij}$ - *Local Balance Equation*

Reversible MC:
A MC is *reversible* wrt a stationary distbn if the Local Balance Eqn holds.
If a MC is reversible wrt $\{\pi_i\}_{i \in S}$ then $\{\pi_i\}_{i \in S}$ is a stationary distbn.
If $\{X_n\}_{n \in \mathbb{Z}} = \{\ldots, X_{-n}, X_{-n+1}, \ldots, X_0, \ldots, X_n, \ldots\}$ is a MC
then $\{Y_n\}$ $with\ Y_n = X_{-n}$ is a MC, and is called the *reversed process*

Chapter 5

Exponential Distribution

$x \sim e^{\lambda} \Rightarrow$ pdf: $f_X(x) = \lambda e^{-\lambda x} \quad (\forall x > 0)$
$F(x) = \int_{-\infty}^{x} f(x)dx = 1 - e^{-\lambda x}$
$E[X] = \int_{\mathbb{R}} xf_X(x)dx = \int_0^{\infty} \lambda xe^{-\lambda x}dx = \frac{1}{\lambda}$
$Var[X] = \frac{1}{\lambda^2}$
$P(X > t) = \int_t^{\infty} f_X(x)dx = e^{(-\lambda t)} \quad (t > 0)$

Properties of Exponential

Memoryless - $P\{X > s + t|X > t\} = P\{X > s\}$ Sums - if
$\{X_i\} \sim exp(\lambda_i)$, then $min(X_1, \ldots, X_n) \sim e^{\lambda_1 + \cdots + \lambda_n}$
and $P(min(X_1, \ldots, X_n) = X_i) = \frac{\lambda_i}{\lambda_1 + \cdots + \lambda_n}$

Gamma Distribution

$\{X_i\}$ drawn idd $\sim exp(\lambda) \Rightarrow X_1 + \ldots + X_n$is gamma$(n, \lambda)$
$f(t) = \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}$

Comparing Exponentials

$P\{X_1 < X_2\} = \int_0^{\infty} P\{X_1 < X_2|X_1 = x\}\lambda_1 e^{-\lambda_1 x}dx =$
$\int_0^{\infty} P\{x < X_2\}\lambda_1 e^{-\lambda_1 x}dx = \frac{\lambda_1}{\lambda_1 + \lambda_2}$

Counting Process

$\{N(t), t \geq 0\}$ is a counting process if:

- $N(t) > 0$
- $N(t)$ is integer valued
- If $s < t$ then $N(s) \leq N(t)$
- For $s < t$, $N(t) - N(s)$ equals the number of events that occur in the interval $(s, t]$

Independent Increments

Increments are independent if *all* two increments with disjoint time intervals are independent.

Poisson Process

Definition 1:
A counting process $\{N(t), t \geq 0\}$ is a Poisson process with rate λ if:

- $N(0) = 0$
- $N(t)$ has stationary, independent increments
- N(t) $\sim$ Poisson (λt), $t > s$:
  $\Rightarrow P\{N(t+s) - N(s) = n\} = e^{-\lambda t}\frac{(\lambda t)^n}{n!}, n = 0, 1, \ldots$

Definition 2:

- $N(0) = 0$
- $N(t)$ has stationary, independent increments
- $P(N(h) = 0) = 1 - \lambda h + O(h)$
  $P(N(h) = 1) = \lambda h + O(h)$
  $P(N(h) \geq 2) = O(h)$

Definition 3:

- $N(0) = 0$
- $N(t)$ = the number of events before time t
- The time between events are i.i.d. $exp(\lambda)$

$O(h)$ Functions

A function $f(\dot{})$ is $o(h)$ if $\lim_{h \to \infty} \frac{f(h)}{h} = 0$

Poisson Properties

$E[N(t)] = \lambda t$
On a unit time interval, on average λ events occur.
$N(t) \to \infty ast \to \infty$
The state space is $S = \mathbb{N} \cup \{0\}$
A Poisson process has non-decreasing paths.

- If $\{N_1(t)\}_{t \geq 0}$ and $\{N_2(t)\}_{t \geq 0}$ are two independent Poisson processes with rates λ_1, λ_2 then
  $\Rightarrow \{N(t)\}_{t \geq 0} = N_1(t) + N_2(t)\}_{t \geq 0}$ is also a Poisson process with rate $\lambda = \lambda_1 + \lambda_2$
  (this is called the *superposition* of $N_1(t)$ and $N_2(t)$).
- $N(0) = N_1(0) + N_2(0) = 0$
- $I(t_1, t_2) = N(t_2) - N(t_1) = N(t_2) - N(t_1) + N_2(t_2) - N_2(t_1)$
  $I(t_1, t_2) \sim$ Poisson $(\lambda_1 + \lambda_2, t_2 - t_1) = N$ has stationary increments.

These three properties $\Rightarrow N \sim$ Poisson

Thinning a Poisson Process:
Take $\{N(t)\}_{t \geq 0}$ with rate λ. Mark each event with probability p independent from event to event. Let $\{N_1(t)\}_{t \geq 0}$ be the process that counts the marker events. Then $\{N_1(t)\}$ is a Poisson process with rate λp.

Bernoulli Process as a Discrete Poisson

$\{X_n\}_{n \geq 1} \sim$ i.i.d. Bernoulli R.V.
$P(X_n = 1) = p; P(X_n = 0) = (1-p); p = \lambda h$
$X_n = 1$ if an arrival occurs in $[(n-1)h, nh]$
Then $B^h(nh) = \sum_{i=1}^n X_i$ - number of arrivals up to time nh
$\{B^h(nh)\}_{n \geq 0}$ - discrete-time process with stationary, independent increments
$\rightarrow$ called a Bernoulli Process
Then: $P(B^h(nh) = l) = \binom{n}{l}(\lambda h)^l(1 - \lambda h)^{n-l}$
And as $h_k \to 0, n_k \to 0$ as $k \to \infty$, $then\ n_k h_k \leq t \leq (n_k + 1)h_k$ so $n_k h_k \to t\ as\ k \to \infty$
So: $\lim_{k \to \infty} P(B^{h_k}(n_k h_k) = l) = P(Poisson(\lambda t) = l)$

Interarrival Time

$T_n$is the time between the $(n-1)^{st}$ and n^{th} events. $\{T_n\}$ is the **sequence of internarrival times** with $T_n \sim$ exp(λ)

Waiting Time

$S_n = \sum_{i=1}^n T_i$ ~gamma(n,λ)

Chapter 6

Continuous Time Markkov Chain

$\{X(t), t \geq 0\} \forall s, t \geq 0$, non-neg ints $i, j, x(u), 0 \leq u < s$ has
$P\{X(t+s) = j|X(s) = i, X(u) = x(u), 0 \leq u \leq s\}$
$= P\{X(t+s) = j|X(s) = i\}$

Stationary Transition Probabilities

A CTMC has these if $P\{X(t+s)=j|X(s)=i\}$

CTMC Alternate Definition

A stochastic process having these properties each time it enters state i:

- The amount of time it spends in that state before make a transition into a different state $\sim exp(\lambda)$ with mean $1/v_i$
- when the process leaves state i, it enters state j with some probability P_{ij} and
  $P_{ii}=0$ all i
  $\sum_j P_{ij}=1$, all i

Birth and Death Processes

A system with n people with

- $\{\lambda_n\}_{n=0}^{\infty}$ the arrival/birth rate
- $\{\mu_n\}_{n=0}^{\infty}$ the departure/death rate
- $v_0=\lambda_0$
- $v_i=\lambda_i+\mu_i$
- $P_{01}=1$
- $P_{i,i+1}=\frac{\lambda_i}{\lambda_i+\mu_i}$
- $P_{i,i-1}=\frac{\mu_i}{\lambda_i+\mu_i}$
- $q_{i,i+1}=\lambda_i$
- $q_{i,i-i}=\mu_i$
- $q_{ii}=0$
- Knowing q's can give us P, but knowing P can't give us q's

Transition Probability Function

Move from state i to state j in a time t later.
$P_{ij}=P(X(t)=j|X(0)=i)$ a continuous function.

Instantaneous Transition Rates

$P_{ij}(t)=P\{X(t+s)=j|X(s)=i\}$
$q_{ij}=v_iP_{ij}$ is the rate, when in state i, at which the process makes a transition into state j.
$v_i=\sum_j v_iP_{ij}=\sum_j q_{ij}$
$P_{ij}=\frac{q_{ij}}{v_i}=\frac{q_{ij}}{\sum_j q_{ij}}$
T_i is the holding time in state i $\sim exp(-v_i)$
$P(T_i>h)=e^{v_ih}$

Chapman-Kolmogorov Equations

$P_{ij}(t+s)=\sum_{k=0}^{\infty}P_{ik}(t)P_{kj}(s),\forall s,t\geq 0$

Kolmogorov's Backward Equations

$P'_{ij}(t)=\sum_{k\neq i}q_{ik}P_{kj}(t)-v_iP_{ij}(t)$

Kolmogorov's Forward Equations

$P'_{ij}(t)=\sum_{k\neq j}q_{kj}P_{ik}(t)-v_jP_{ij}(t)$

Limiting Probabilities

$P_j\equiv\lim_{t\to\infty}P_{ij}(t)$ The limiting probs will exist if

- all states communicate
- the chain is positive recurrent

Use that to get:
$v_jP_i=\sum_{k\neq j}q_{kj}P_k$, all states j
(leaving = entering)
$\sum_j P_j=1$ In the *discrete* case π may exists but limiting probabilities may not (if discrete chain is not aperiodic)
In the *continuous* case there is no similar problem, if π exists it is unique a and the above limit holds.

Embeded Chains

$\{X(t)\}_{t\geq 0}$ a CTMC
π - stationary distribution of X
ψ - stationary distribution of the embedded discrete time MC
$\psi_j=\frac{\pi_jv_j}{\sum_{i\in S}\pi_iv_i}$, $\pi_j=\frac{\psi_j/v_j}{\sum_{i\in S}\psi_i/v_i}$
ψ_j - long run proportoin of transitions that CTMC makes into state j
$1/v_J$ - average time it stays in state j
ψ_j/v_j - long run proportion of time the CTMC spends in state j
Note: ψ may exist and π may not!

Local Balance Equation for CTMC

$\pi_iq_{ij}=\pi_jq_{ji}$ means rate of flow from i to j = rate of flow from j to i

Time Reversability

For a long running MC, the amount of time the process spends in state i is also exponentially distributed with rate v_i. We have the discrete time reversed chain:
$Q_{ij}=\frac{\pi_jP_{ji}}{\pi_i}$ and then
$\pi_iP_{ij}=\pi_jP_{ji}\forall i,j$
$P_iq_{ij}=P_jq_{ji}\forall i,j$

Chapter 8

Definitions

L	average # of customers in system
L_Q	average # of customers waiting in queue
W	average amount of time a customer spends in system
W_Q	average amount of time a customer spends in queue
$E[S]$	average amount of time customer spends in service
$N(t)$	number of customer arrivals by time t
P_n	number of customers in system at time t $=\lim_{t\to\infty}P\{X(t)=n\}$ • aka limiting/longrun/steady state probability that n customers are in the system • also long run proportion of time that the system contains n customers
λ_a	average arrival rate of customers $=\lim_{t\to\infty}\frac{N(t)}{t}$
a_n	proportion of custs that find n in the system when arriving
d_n	proportion of custs that find n in the system when leaving

Littile's Formula

$L=\lambda_aW$ $L_Q=\lambda_aW_Q$

Poisson Model

Poisson arrivals see time averages. ie $P_n=a_n$.

M/M/1

Customers arrive according to a Poisson process with rate λ. The time between successive arrivals are independent rv with mean $1/\lambda$. If server free, cust goes in, else into queue. Service times are $\sim$exp(μ).
Balance Equations:
$\lambda P_0=\mu P_1$
$(\lambda+\mu)P_n=\lambda P_{n-1}+\mu P_{n+1}$
$P_1=\frac{\lambda}{\mu}P_0$
$P_{n+1}=\frac{\lambda}{\mu}P_n+(P_n-\frac{\lambda}{\mu}P_{n-1})=\left(\frac{\lambda}{\mu}\right)^{n+1}P_0$
$1=\sum_{n=0}^{\infty}P_n=\sum_{n=0}^{\infty}\left(\frac{\lambda}{\mu}\right)^nP_0=\frac{P_o}{1-\lambda/\mu}$ or
$\Rightarrow P_0=1-\frac{\lambda}{\mu}, P_n=\left(\frac{\lambda}{\mu}\right)^n(1-\frac{\lambda}{\mu})$
$L=\sum_{n=0}^{\infty}nP_n=\frac{\lambda}{\mu-\lambda}$
$W=\frac{L}{\lambda}=\frac{1}{\mu-\lambda}$
$W_Q=W-E[S]=W-1/\mu=\frac{\lambda}{\mu(\mu-\lambda)}$
$L_Q=\lambda W_Q=\frac{\lambda^2}{\mu(\mu-\lambda)}$

M/M/1 - Finite Capacity

Now we have the limitation that $n\leq N$.
Balance Equations:
$\lambda P_0=\mu P_1$
$(\lambda+\mu)P_n=\lambda P_{n-1}+\mu P_{n+1}, 1\leq n\leq N-1$
$\mu P_n=\lambda P_{n-1}$, for state N
$P_1=\frac{\lambda}{\mu}P_0$
$P_{n+1}=\frac{\lambda}{\mu}P_n+(P_n-\frac{\lambda}{\mu}P_{n-1}), 1\leq n\leq N-1$
$P_N=\frac{\lambda}{\mu}P_{N-1}\dots=\left(\frac{\lambda}{\mu}\right)^NP_0$
$1=\sum_{n=0}^{\infty}P_n=P_0\sum_{n=0}^{\infty}\left(\frac{\lambda}{\mu}\right)^n=P_0\frac{1-(\lambda/\mu)^{N+1})}{1-\lambda/\mu}$ or
$\Rightarrow P_0=\frac{1-\lambda/\mu}{1-(\lambda/\mu)^{N+1}}, P_n=\frac{(\lambda/\mu)^n(1-\lambda/\mu)}{1-(\lambda/\mu)^{N+1}}, n=0,\dots,N$
$L=\sum_{n=0}^{N}nP_n=\frac{(1-\lambda/\mu)}{1-(\lambda/\mu)^{N+1}}\sum_{n=0}^{N}n\left(\frac{\lambda}{\mu}\right)^n=$
$=\frac{\lambda[1+N(\lambda/\mu)^{N+1}-(N+1)(\lambda/\mu)^N]}{(\mu-\lambda)(1-(\lambda/\mu)^{N+1})}$
To find W we consider 2 cases.
$\lambda_a=\lambda$ if "customers in system" includes those who never get in
$\lambda_a=\lambda(1-P_N)$ if it does not. Either way we get:
$W=\frac{L}{\lambda_a}$

PASTA

Poisson Arrivals See Time Averages
Let $\{X_i\}_{t\geq 0}$ a continuous-time Markov chain with stationary distribution π. Let T_i be the arrival time of the i^{th} element. These elements arrive according to a Poisson process.
Then $\{X(T_n)\}_{t\geq 0}$ has π as a stationary distribution.
$a_j=\pi_j$

Other useful stuff

$\sum_{n=0}^{\infty}nx^n=\frac{x}{(1-x)^2}$
$X\sim$ Bernoulli(p) means that you have an even with probability of success p.
$X\sim$ geometric(p) means X is the number of Bernoulli trials until success.
  $p(n)=p\{X=n\}=(1-o)^{n-1}p$
$X\sim$ binomial(n,p) X is the number of successes in n trials.
  $p(i)=\binom{n}{i}p^i(1-p)^{n-i}$
$\binom{n}{i}=\frac{n!}{(n-i)!i!}$
$\sum_{i=0}^{N}\binom{N}{i}=(1+1)^N=2^N$
$\sum_{i=1}^{k}m^i=\frac{m(1-m^k)}{1-m}$

Infitesimal Generator: G

$g_{ij}=q_{ij}, i\neq j$
$g_{ii}=-v_i$ (note: on last 2 lines, entries not probabilities
$[P\prime(t)]_{ij}=[GP(t)]_{ij}$
$P\prime(t)=GP(t)$(K's backwards eqn)
$P\prime(t)=P(t)G$ (K's forward eqn)
$P(t)=e^{tg}=\sum_{n=0}^{\infty}\frac{(tG)^n}{n!}$
a

Stationary Distribution for a CTMC

$\pi=\pi P(t)$
$\sum\pi=1$
If the initial distribution of $X(0)$ is π then the distribution of $X(t)$ will also be π, $t>0$. ($P(X(t)=j)=\pi_j\forall j\in S, t>0$)
$\sum_{i\in S}p_{ij}(t)\pi_i=\pi_j$

Global balance equation for CTMC

$0=\pi G$
$v_j\pi_j=\sum_{i\neq j}q_{ij}\pi_i$ this is the jth row of the matrix
$\Rightarrow$ long run rate out of j = long run rate into state i
π_i long run proportion of time that process in in state j.
v_j rate of leaving state j
π_jv_j long run rate of leaving j
π_iq_{ij} long run rate of going from state i to j